tate facilè perſpecturus foret; jam vero, quoniam egregium illud Rei Medicæ Lumen amiſimus, eadem aliis Eruditis perpendenda ſimul proponimus & dijudicanda. Tibi præſertim, Vir Doctiſſime, cujus auctoritatem & ille plurimi fecit, & nos præcipuam habemus, Judici ſimul integerrimo & maxime idoneo, totam iſtam diſputationem lubentiſſime ſubjicimus.

III. *Methodus Differentialis* Newtoniana *Illuſtrata. Authore* Jacobo Stirling, *è Coll. Balliol.* Oxon.

ARithmeticæ pars præcipua conſiſtit in inveniendâ in numeris quantitate quâcunque determinatâ; cum vero quantitatum & numerorum natura non patiatur ut omnes quantitates exhibeantur in numeris accurate, neceſſe habemus ad Approximationes confugere. Hoc eſt, ubi quantitatum valores mathematice accurati nequeunt obtineri, quærendi ſunt ii qui ab accuratis diſtant minus datâ quâvis differentiâ.

Quicquid hâc de re à Veteribus ad nos pervenit, vel eſt particulare, ut Methodus eorum reducendi Æquationes Quadraticas; vel ſaltem uſibus generalibus male deſtinatum, ut Methodus Exhauſtionum. *Vieta* quidem primus erat qui aliquid generale in hâc arte aſſequutus eſt: quippe invenit methodum reducendi Æquationes Rationales, quæ ſolæ tunc in uſu erant. In hâc acquievêre omnes Geometræ ex ejus temporibus uſque ad ea *Newtoni*. Hic ex Interpolationibus primo pervenit ad Series: quas poſtea ad reductionem Æquationum omnium omnino generum univerſaliter applicuit. Hæc autem methodus procedit per quantitatum naſcentium & evaneſcentium rationes primas & ultimas, ſeu ſi ita loqui liceat, per quantitatum coincidentium

cidentium differentias infinite parvas. Sed & ulterius promovit *Newtonus* hanc methodum; docuitque qua ratione approximandum sit ad quantitates quæ determinantur per regularem seriem terminorum, non per Æquationem ut vulgo fit. Atque sic posuit fundamenta calculi hujus Differentialis, qui procedit per quantitatum differentias cujuscunque magnitudinis: ideoque est methodo Serierum universalior. Per hasce artes *Newtonianas* universa doctrina Approximationum reducitur ad solutionem Problematis, *Invenire Lineam Geometricam quæ per data quotcunque puncta transibit.* Ex hujus inquam solutione inveniuntur radices Æquationum quarumcunque, & etiam quantitates quarum relationes ad alias datas per nullas Æquationes hactenus notas possunt exprimi. Existimo igitur *Newtonum* perduxisse methodum Approximandi ad summum perfectionis fastigium; dum ex unico simplicissimo principio totam hanc doctrinam longe lateque patentem deducit. Quapropter credendum est animum *Newtoni* non satis perspectum fuisse iis, qui ejus methodos appellant particulares, & alias tanquam suas & solas genuinas atque generales venditant, quæ aliæ non erant quam Corollaria facillima à *Newtonianis.*

Author noster, in Epistola ad *Oldenburgum, Octob.* 24. 1676. data, mentionem fecit de methodo expeditâ ducendi Lineam Parabolicam per data quotcunque puncta; qua dixit se usum fuisse ubi Series simplices non sunt satis tractabiles. Et hanc methodum primo publicavit in Lemmate quinto Libri tertii *Principiorum.* Atque in Lectionibus publicis, circa idem tempus quo dicta Epistola scripta est, *Cantabrigiæ* habitis, exposuit modum generalem determinandi Curvas cujuscunque generis quæ transibunt per totidem data puncta quot earum natura patitur. Hæ Lectiones sub titulo *Arithmeticæ Universalis* anno 1707. publicatæ sunt, ubi habetur

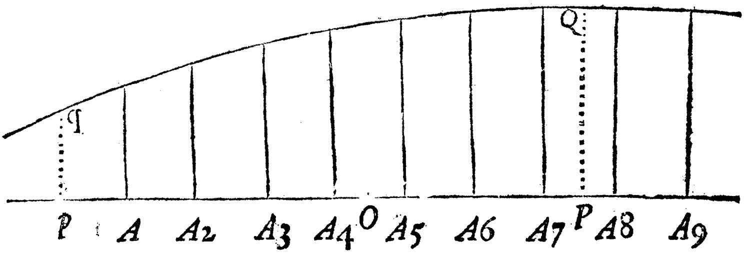
betur methodus exemplis illustrata in sectionibus Conicis. Anno vero 1711. tandem prodiit, inter alios ejusdem Authoris tractatus, ipsa Methodus Differentialis plenius quam ante exposita, cum fundamento ejus demonstrato

Archimedes in methodo Exhaustionum, *Cavallerius* in methodo Indivisibilium, & *Wallisius* noster in Arithmetica Infinitorum, posuerunt fundamenta doctrinæ de determinanda quantitate quæsitâ per locum quem obtinet inter terminos in data Serie: at qua ratione approximandum esset ad valores quantitatum sic determinatarum. horum nemo docuit; Hoc primus & solus perfecit *Newtonus:* atque exinde haud parum ampliata est universa Analysis. Nam sicut ante hoc inventum, ea Problemata Arithmetica sola pro solutis habebantur, ubi relatio quantitatis quæsitæ ad alias datas definiebatur Æquatione, jam pro solutis habenda sunt non minus ea, in quibus quantitas quæsita locum datum sortitur inter terminos datæ Seriei; siquidem numeri desiderati non minus accurate obtinentur per Methodum Differentialem, quam per extractionem Radicum: hisce vero habitis, parum interest quomodo ad eos deventum est. Et experientia multiplex docuit, quod plurima Problemata ad Æquationes ægre deducuntur, dum ad methodum Differentialem facillime. Qualis est ex multis aliis toties decantata Circuli Quadratura; quam tam perfectam, mea opinione, *Wallisius* in Arithmetica Infinitorum exhibuit quam *Archimedes* illam Parabolæ.

Propositio

Propositio.

Invenire Lineam Parabolicam quæ transibit per extrema Ordinatorum quotcunque æquidistantium.



B B2 B3 B4 B5 B6 B7 B8

C C2 C3 C4 C5 C6 C7

D D2 D3 D4 D5 D6

E E2 E3 E4 E5

F F2 F3 F4

G G2 G3

H H2

I

Casus Primus.

Designent A, $A2$, $A3$, $A4$, $A5$, $A6$, $A7$, $A8$, $A9$, *&c.* Ordinatas æquidistantes insistentes Abscissæ in dato angulo. Collige earum differentias B, $B2$, $B3$, $B4$, $B5$, $B6$, $B7$, $B8$, *&c.* harumque differentias C, $C2$, $C3$, $C4$, $C5$, $C6$, $C7$, *&c.* harumque differentias D, $D2$, $D3$, $D4$, $D5$, $D6$, *&c.* harumque differentias E, $E2$, $E3$, $E4$, $E5$, *&c.* harumque F, $F2$, $F3$, $F4$, *&c.* Et sic porro. Differentiæ autem colligi debent auferendo

ferendo priores semper de posterioribus. Hoc est ponendo $B = A2 - A$, $B2 = A3 - A2$, $B3 = A4 - A3$, $B4 = A5 - A4$, $B5 = A6 - A5$, &c. Tum $C = B2 - B$, $C2 = B3 - B2$, $C3 = B4 - B3$, $C4 = B5 - B4$, &c. deinde $D = C2 - C$, $D2 = C3 - C2$, $D3 = C4 - C3$, &c. Et similiter sunt omnes differentiæ sequentes colligendæ. Vel sint $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta$, &c. æquales A, $A2$, $A3$, $A4$, $A5$, $A6$, $A7$, &c. Eritque $A = \alpha$, $B = \beta - \alpha$, $C = \gamma - 2\beta + \alpha$, $D = \delta - 3\gamma + 3\beta - \alpha$, $E = \varepsilon - 4\delta + 6\gamma - 4\beta + \alpha$, $F = \zeta - 5\varepsilon + 10\delta - 10\gamma + 5\beta - \alpha$, $G = \eta - 6\zeta + 15\varepsilon - 20\delta + 15\gamma - 6\beta + \alpha$, &c. In hisce valoribus numerales Coefficientes ipsorum $\alpha, \beta, \gamma, \delta, \varepsilon$ &c. generantur ut in dignitatibus integris Binomii $\overline{1-z}|^0$, $\overline{1-z}|^1$, $\overline{1-z}|^2$, $\overline{1-z}|^3$, $\overline{1-z}|^4$, &c. Scribendo numeros 1, 2, 3, 4, 5, &c. in Serie $1 \times \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} \times$ &c. successive pro n. Sit jam PQ quælibet Ordinata reliquis intermedia, & AP ejus distantia ab Ordinata prima A appelletur z, tum erit

$$
\begin{aligned}
PQ = A &+ \\
&B \times \frac{z}{1} + \\
&C \times \frac{z}{1} \times \frac{z-1}{2} + \\
&D \times \frac{z}{1} \times \frac{z-1}{2} \times \frac{z-2}{3} + \\
&E \times \frac{z}{1} \times \frac{z-1}{2} \times \frac{z-2}{3} \times \frac{z-3}{4} + \\
&F \times \frac{z}{1} \times \frac{z-1}{2} \times \frac{z-2}{3} \times \frac{z-3}{4} \times \frac{z-4}{5} + \\
&G \times \frac{z}{1} \times \frac{z-1}{2} \times \frac{z-2}{3} \times \frac{z-3}{4} \times \frac{z-4}{5} \times \frac{z-5}{6} + \text{\&c.}
\end{aligned}
$$

Adeoque

Adeoque signum ipsius z mutandum est, quando PQ cadit ad alteras partes Ordinatæ primæ, ut pq.

Casus Secundus.

Sit jam $A5$ Ordinata in medio omnium; pone $A = B4 + B5$, $B = D3 + D4$, $C = F2 + F3$, $D = H + H2$, &c. & $a = C4$, $b = E3$, $c = G2$, $d = I$, &c. id est, si sint $A6 = \alpha$, $A7 = \beta$, $A8 = \gamma$, $A9 = \delta$, &c. $A4 = \varkappa$, $A3 = \lambda$, $A2 = \mu$, $A = \nu$, &c. Pone $A = \alpha - \varkappa$, $B = \beta - 2\alpha + 2\varkappa - \lambda$, $C = \gamma - 4\beta + 5\alpha - 5\varkappa + 4\lambda - \mu$, $D = \delta - 6\gamma + 14\beta - 14\alpha + 14\varkappa - 14\lambda + 6\mu - \nu$, &c. $a = \alpha - 2A5 + \varkappa$, $b = \beta - 4\alpha + 6A5 - 4\varkappa + \lambda$, $c = \gamma - 6\beta + 15\alpha - 20A5 + 15\varkappa - 6\lambda + \mu$, $d = \delta - 8\gamma + 28\beta - 56\alpha + 70A5 - 56\varkappa + 28\lambda - 8\mu + \nu$, &c. Et dicatur $A5P$, z, tum erit

$$PQ = A5 + \frac{Az + azz}{1.2} +$$
$$\frac{2Bz + bzz}{1.2} \times \frac{zz-1}{3.4} +$$
$$\frac{3Cz + czz}{1.2} \times \frac{zz-1}{3.4} \times \frac{zz-4}{5.6} +$$
$$\frac{4Dz + dzz}{1.2} \times \frac{zz-1}{3.4} \times \frac{zz-4}{5.6} \times \frac{zz-9}{7.8} +$$
$$\frac{5Ez + ezz}{1.2} \times \frac{zz-1}{3.4} \times \frac{zz-4}{5.6} \times \frac{zz-9}{7.8} \times \frac{zz-16}{9.10} +$$
&c.

Casus Tertius.

Sint jam $A4$, $A5$, Ordinatæ duæ in medio omnium: Pone $A = \frac{A4 + A5}{2}$, $B = \frac{C3 + C4}{2}$, $C = \frac{E2 + E3}{2}$, $D =$

$\frac{G+G2}{2}$, &c. $a=B4$, $b=D3$, $c=F2$, $d=H$, &c. Vel sint $A5=\alpha$, $A6=\beta$, $A7=\gamma$, $A8=\delta$, &c. $A4=\varkappa$, $A3=\lambda$, $A2=\mu$, $A=\nu$, &c. Deinde erunt $2A=\alpha+\varkappa$, $2B=\beta-\alpha-\varkappa+\lambda$, $2C=\gamma-3\beta+2\alpha+2\varkappa-3\lambda+\mu$, $2D=\delta-5\gamma+9\beta-5\alpha-5\varkappa+9\lambda-5\mu+\nu$, &c. Et $a=\alpha-\varkappa$, $b=\beta-3\alpha+3\varkappa-\lambda$, $c=\gamma-5\beta+10\alpha-10\varkappa+5\lambda-\mu$, $d=\delta-7\gamma+21\beta-35\alpha+35\varkappa-21\lambda+7\mu-\nu$, &c. Et sit O punctum medium inter $A4$, $A5$, atque appelletur OP, z; eritque Ordinata

$$PQ=\frac{A+az}{4^0}+$$
$$\frac{3B+bz}{4^1}\times\frac{4zz-1}{2\cdot 3}+$$
$$\frac{5C+cz}{4^2}\times\frac{4zz-1}{2\cdot 3}\times\frac{4zz-9}{4\cdot 5}+$$
$$\frac{7D+dz}{4^3}\times\frac{4zz-1}{2\cdot 3}\times\frac{4zz-9}{4\cdot 5}\times\frac{4zz-25}{6\cdot 7}+$$
$$\frac{9E+ez}{4^4}\times\frac{4zz-1}{2\cdot 3}\times\frac{4zz-9}{4\cdot 5}\times\frac{4zz-25}{6\cdot 7}\times\frac{4zz-49}{8\cdot 9}+\text{\&c.}$$

In hisce duobus etiam casibus z est negativa, quando Ordinata PQ cadit ad alteras partes initii Abscissæ. Et in omnibus tribus casibus distantia communis Ordinatarum ponitur unitas.

Omnes tres casus demonstrantur facillime per calculum. In casu primo pro PQ scribo successive $\alpha, \beta, \gamma, \delta, \varepsilon$, &c. & pro z interea 0, 1, 2, 3, 4, &c. quæ sunt longitudines Abscissæ ordine sequentes; & provenient æquationes

$$\alpha=A,\ \beta=A+B,\ \gamma=A+2B+C,\ \delta=A+3B+3C+D,$$
$$\varepsilon=A+4B+6C+4D+E,\ \text{\&c.}$$

$\beta-\alpha$

$\beta-\alpha=B$, $\gamma-\beta=B+C$, $\delta-\gamma=B+2C+D$, $\varepsilon-\delta=B+3C+3D+E$, &c.

$\gamma-2\beta+\alpha=C$, $\delta-2\gamma+\beta=C+D$, $\varepsilon-2\delta+\gamma=C+2D+E$, &c.

$\delta-3\gamma+3\beta-\alpha=D$, $\varepsilon-3\delta+3\gamma-\beta=D+E$, &c.

$\varepsilon-4\delta+6\gamma-4\beta+\alpha=E$, &c.

Hæ Æquationes, capiendo earum differentias, nullo labore resolvuntur, uti videre est. Et dant eosdem ipsorum A, B, C, D, &c. valores, qui antea positi sunt in solutione. Et ad eundem modum demonstrantur casus duo reliqui.

Harum trium serierum unaquæque converget ad valorem Ordinatæ PQ, ubi Ordinatarum datarum differentiæ sunt justæ magnitudinis. At ubi non convergunt, aliæ artes adhibendæ sunt. Sed impræsentiarum de hujus Propositionis usu pauca adjiciamus.

Designent α, β, γ, δ, ε, ζ, η, θ, $\varkappa$, λ, &c. terminos quoscunque æquidistantes, quorum differentiæ sunt perexiguæ; & relationes quas inter se obtinent definientur quamproxime per Æquationes sequentes, quæ oriuntur capiendo differentias & differentias differentiarum continuò, & ponendo eas æquales nihilo.

$$\alpha-\beta=0$$
$$\alpha-2\beta+\gamma=0$$
$$\alpha-3\beta+3\gamma-\delta=0$$
$$\alpha-4\beta+6\gamma-4\delta+\varepsilon=0$$
$$\alpha-5\beta+10\gamma-10\delta+5\varepsilon-\zeta=0$$
$$\alpha-6\beta+15\gamma-20\delta+15\varepsilon-6\zeta+\eta=0$$
$$\alpha-7\beta+21\gamma-35\delta+35\varepsilon-21\zeta+7\eta-\theta=0$$
$$\alpha-8\beta+28\gamma-56\delta+70\varepsilon-56\zeta+28\eta-8\theta+\varkappa=0$$
$$\alpha-9\beta+36\gamma-84\delta+126\varepsilon-126\zeta+84\eta-36\theta+9\varkappa-\lambda=0.$$

&c.

Hæc

Hæc Tabula in usum reservanda est, ut consulatur quoties opus sit. Quod autem hæ Æquationes vel obtinent accurate, vel ad verum approximant, ubi differentiæ terminorum sunt parvæ, patet ex demonstratione casus primi Propositionis.

Assumatur quælibet Series $\frac{1}{101}$, $\frac{1}{102}$, $\frac{1}{103}$, $\frac{1}{104}$, $\frac{1}{105}$, $\frac{1}{106}$, &c. Et quæratur terminus qui stat proximus ante $\frac{1}{101}$: patet quod ille est $\frac{1}{100}$; videamus ergo qualem hæc methodus exhibebit eundem. Repræsentet α terminum quæsitum, eritque

$\frac{1}{101} = \beta = 0099{,}0099{,}0099{,}0$,	Æquatio.	1ma	dat α	0099,0099,0099,0,
$\frac{1}{102} = \gamma = 0098{,}0392{,}1568{,}7$,		2da		0099,9805,8629,3,
$\frac{1}{103} = \delta = 0097{,}0873{,}7864{,}1$,		3tia		0099,9994,3455,0,
$\frac{1}{104} = \varepsilon = 0096{,}1538{,}4615{,}4$,		4ta		0099,9999,7824,8,
$\frac{1}{105} = \zeta = 0095{,}2380{,}9523{,}8$,		5ta		0099,9999,9895,8,
$\frac{1}{106} = \eta = 0094{,}3396{,}2264{,}2$.		6ta		0099,9999,9993,1.

Patet ergo quod hæc methodus continue approximat. Si terminorum differentiæ fuissent minores, valores accessissent citius ad verum, & contra tardius quando differentiæ sunt majores. Hinc si in Tabulis numericis desit terminus, potest is per hanc methodum inseri.

Hoc modo etiam prodeunt ipsissimæ Series Speciosæ, quæ per alias methodos prodire solent. Proponatur $\overline{1+zz}|^{-1}$ Ordinata Curvæ quadrandæ: Ea est prima in serie regulari $\overline{1+zz}|^{-1}$, $\overline{1+zz}|^{0}$, $\overline{1+zz}|^{1}$, $\overline{1+zz}|^{2}$, $\overline{1+zz}|^{3}$, &c. Ordinatarum, quæ omnes præter primam dant suas areas z, $z+\frac{1}{3}z^3$, $z+\frac{2}{3}z^3+\frac{1}{5}z^5$, $z+\frac{3}{3}z^3+\frac{3}{5}z^5+\frac{1}{7}z^7$, &c. constituentes novam seriem cujus primus terminus erit Area quæsita: quæ ideo invenietur ponendo pro eâ α, & pro reliquis in suo Ordine β, γ, δ, ε, &c. Prima Æquatio dat $\alpha = z$, secunda $\alpha = z - \frac{1}{3}z^3$, tertia $\alpha = z - \frac{1}{3}z^3 + \frac{1}{5}z^5$, quarta $\alpha = z - \frac{1}{3}z^3 + \frac{1}{5}z^5 - \frac{1}{7}z^7$, &c.

&c. Eſt ergo univerſim area quæſita $z - \frac{1}{3}z^3 + \frac{1}{5}z^5 - \frac{1}{7}z^7 + \frac{1}{9}z^9 - \frac{1}{11}z^{11}$ &c. Eſtque hæc Series arcus ad Tangentem z, in circulo radium habente unitati æqualem. Eam invenit *Jacobus Gregorius* noſter, & cum *Collinio* communicavit initio anni 1671. à quo, mediante *Oldenburgo* ad *Leibnitium* delata eſt.

Sit jam &c, $e, d, c, b, a, P, \alpha, \beta, \gamma, \delta, \varepsilon,$ &c. Series utrinque excurrens in infinitum, ubi dantur omnes termini præter P in medio omnium. Sit $A = \alpha + a$, $B = \beta + b$, $C = \gamma + c$, $D = \delta + d$, $E = \varepsilon + e$, &c. atque erit

$$P = \frac{A}{2} +$$
$$\frac{A - B}{6} +$$
$$\frac{5A - 8B + 3C}{60} +$$
$$\frac{7A - 14B + 9C - 2D}{140} +$$
$$\frac{42A - 96B + 81C - 32D + 5E}{1260} +$$
$$\frac{66A - 165B + 165C - 88D + 25E - 3F}{2772} +$$
$$\frac{429A - 1144B + 1287C - 832D + 325E - 72F + 7G}{24024} + \&c.$$

Inveſtigatur hæc Series ex Æquationibus, excerpendo alternas in quibus numerus terminorum eſt impar. Nam earum differentiæ relinquent terminos in hac Serie; quæ itaque ad libitum produci poteſt.

Sit $\overline{1+z}|^{-1}$ Ordinata Hyperbolæ, & quæratur Area ejus quæ jacet ſupra Abſciſſam z, quando ea evadit unitas. Hæc Ordinata eſt media in Serie Ordinatarum

rum, &c. $\overline{1+z}|^{-5}$, $\overline{1+z}|^{-4}$, $\overline{1+z}|^{-3}$, $\overline{1+z}|^{-2}$, $\overline{1+z}|^{-1}$, $\overline{1+z}|^{0}$, $\overline{1+z}|^{1}$, $\overline{1+z}|^{2}$, $\overline{1+z}|^{3}$ &c. æquidiſtantium, hinc inde excurrente in infinitum. Adeoque Areæ ab hiſce Ordinatis genitæ conſtituent ſeriem conſimilem, cujus medius terminus erit Area quæſita; quæ proinde obtinebitur per Seriem modo expoſitam. Quando z eſt unitas, ut in caſu præſente, areæ curvarum evadunt &c. $\frac{15}{64}$, $\frac{7}{24}$, $\frac{3}{8}$, $\frac{1}{2}$, & 1, $\frac{3}{2}$, $\frac{7}{3}$, $\frac{15}{4}$, &c. Hinc eſt $A = 1 + \frac{1}{2} = \frac{3}{2}$, $B = \frac{3}{2} + \frac{3}{8} = \frac{15}{8}$, $C = \frac{7}{3} + \frac{7}{24} = \frac{21}{8}$, $D = \frac{15}{4} + \frac{15}{64} = \frac{255}{64}$, &c. Hiſce in Serie ſubſtitutis, prodit P, id eſt, area Hyperbolæ, $\frac{3}{4} - \frac{3}{48} + \frac{3}{480} - \frac{3}{4480} +$ &c. id eſt, $\frac{3}{4} - \frac{A}{4\cdot 3} - \frac{2B}{4\cdot 5} - \frac{3C}{4\cdot 7} - \frac{4D}{4\cdot 9} - \frac{5E}{4\cdot 11} -$ &c. Ubi jam A, B, C D, &c. more *Newtoniano*, deſignant terminos in ſuo ordine ab initio. Calculum appono.

TERMINI.

Affirmativi	*Negativi.*
7500,0000,0000,0000,0	0625,0000,0000,0000,0
62,5000,0000,0000,0	6,6964,2857,1428,5
7440,4761,9047,6	845,5086,5800,8
97,5586,9130,8	11,3818,4731,9
1,3390,4086,1	1585,7062,8
188,7745,5	22,5708,7
2,7085,0	3260,2
393,4	47,5
5,7	7
+7563,2539,3930,7494,1	—0631,7821,3370,8041,1

Summam negativam ſubducens ab affirmativâ, habeo pro Areâ, id eſt, pro Logarithmo Hyperbolico Binarii, numerum 6931,4718,0559,9453.

Pro

Pro constructione Tabularum quarumvis numericarum percommoda est Series quæ sequitur. Designent &c. $e, d, c, b, a, \alpha, \beta, \gamma, \delta, \varepsilon$, &c. terminos alternos in Serie utrinque serpente in infinitum; Pone $A = \alpha + a$, $B = \beta + b$, $C = \gamma + c$, $D = \delta + d$, $E = \varepsilon + e$, &c. Et terminus inter α & a erit

$$\frac{A}{2} +$$
$$\frac{1}{1} \times \frac{A-B}{2^4} +$$
$$\frac{1 \cdot 3}{1 \cdot 2} \times \frac{2A-3B+C}{2^7} +$$
$$\frac{1 \cdot 3 \cdot 5}{1 \cdot 2 \cdot 3} \times \frac{5A-9B+5C-D}{2^{10}} +$$
$$\frac{1 \cdot 3 \cdot 5 \cdot 7}{1 \cdot 2 \cdot 3 \cdot 4} \times \frac{14A-28B+20C-7D+E}{2^{13}} +$$
$$\frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} \times \frac{42A-90B+75C-35D+9E-F}{2^{16}} +$$
$$\frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \cdot 11}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} \times \frac{132A-297B+275C-154D+54E-11F+G}{2^{19}} +$$
&c.

Hæc Series sequitur ex casu tertio Propositionis, ponendo $z = 0$. Coefficientes numerales literarum sic producuntur; exempli gratiâ, in quarto termino coefficiens literæ penultimæ C est 5; pone $5 + 1 = n$, & numeri qui proveniunt ex multiplicatione terminorum $1 \times \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} \times$ &c. erunt 1, 6, 15, 20, &c. Horum differentiæ 5, 9, 5, sunt numeri quæsiti. Atque adeo Series ad libitum produci potest.

Datis Logarithmis numerorum 46, 48, 50, 52, 54, 56, 58 & 60; invenire Logarithmum numeri 53, qui consistit in medio omnium. Pone $l,52 + l,54 = A = 3,4483,9710,34$, $l,50 + l,56 = B = 3,4471,5803,13$.

$l,48$

$l, 48 + l, 58 = C = 3,4446,6923,08$, $l, 46 + l, 60 = D = 3,4409,0908,19$. Hisce valoribus in Serie scriptis, primi quatuor termini dabunt 1,7242,2586,96 pro Logarithmo numeri 53. Et eadem ratione invenire licet quemvis alium intermedium.

In Constructione ergo Tabularum sufficit primo quærere aliquos terminos in debitis distantiis nam reliqui possunt hoc modo interseri. Etenim continuo sunt intercalandi termini primo inventi usque dum perventum fuerit ad ultimos qui desiderantur. Hoc modo habebitur tota Tabula ex datis paucis terminis sub initio pro fundamento operationis. Sed non convenit ut termini quos primo quærimus, sint omnes per totam Tabulam æquidistantes; nam si omittimus alternos ubi eorum differentia est maxima, possumus alibi per saltum omittere duos, tres, viginti aut forte plures terminos. Numerus autem terminorum inter duos datos consistentium, qui omittuntur, debet semper esse aliquis sequentium 1, 3, 7, 15, 31, 63, &c. dummodo volumus inserere eos per hanc Seriem; hoc vero neutiquam incommodabit opus.

Possunt autem pro Praxi termini in unam summam colligi, ut factum vides in hac Tabella. Prima expressio est primus terminus; secunda est summa primi & secundi; tertia est summa primi, secundi & tertii: & sic porro.

2	$\frac{A}{2}$
4	$\frac{9A-B}{16}$
6	$\frac{150A-25B+3C}{256}$
8	$\frac{1225A-245B+49C-5D}{2048}$
10	$\frac{39690A-8820B+2268C-305D+35E}{65536}$

Sic

Sic datis aliquibus terminis alternis, intermedii confestim dabuntur per hasce expressiones, nullâ ratione habitâ naturæ Tabulæ particularis. Nam hæ regulæ sunt eædem in omnibus. Areæ curvarum sunt proxime æquales areis Parabolicæ figuræ quæ transit per extrema Ordinatarum suarum. Sed quoniam laboriosum nimis esset semper recurrere ad Parabolam, computavi Tabulam sequentem, quâ Areæ directe exhibentur ex datis Ordinatis.

1	$\frac{A}{1}R$
3	$\frac{A+4B}{6}R$
5	$\frac{7A+32B+12C}{90}R$
7	$\frac{41A+216B+27C+272D}{840}R$
9	$\frac{989A+5888B-928C+10496D-4540E}{28350}R$
11	$\frac{16067A+106300B-48525C+272400D-260550E+427368F}{598752}R.$

Hic numerus Ordinatarum est impar, A est summa primæ & ultimæ, B secundæ & penultimæ, C tertiæ & antepenultimæ; & sic porro, usque dum deventum sit ad eam in medio omnium, quæ per ultimam literam in quâque expressione repræsentatur. R est basis seu pars Abscissæ inter primam & ultimam Ordinatam interceptæ. Expressiones sunt Areæ contentæ inter Curvam, basin & Ordinatas hinc inde extremas. Tabulam pro pare numero Ordinatarum non apposui, quoniam Area cæteris paribus ex impare earum numero accuratius definitur.

Quæratur area quæ generatur ab Ordinatâ $\overline{1+zz}|^{-1}$ & jacet supra Abscissam z quando ea evadit unitas. In

$\overline{1+zz}|^{-1}$, pro z scribe $\frac{0}{10}, \frac{1}{10}, \frac{2}{10}, \frac{3}{10}, \frac{4}{10}, \frac{5}{10}, \frac{6}{10}, \frac{7}{10}, \frac{8}{10}, \frac{9}{10}, \frac{10}{10}$; & prodibunt undecim Ordinatæ $1, \frac{100}{101}, \frac{25}{26}, \frac{100}{109}, \frac{25}{29}, \frac{4}{5}, \frac{25}{34}, \frac{100}{149}, \frac{25}{41}, \frac{100}{181}, \frac{1}{2}$. Hinc est $A = 1 + \frac{1}{2} = \frac{3}{2}$, $B = \frac{100}{101} + \frac{100}{181} = \frac{28200}{18281}$, $C = \frac{25}{26} + \frac{25}{41} = \frac{1675}{1066}$, $D = \frac{100}{109} + \frac{100}{149} = \frac{25800}{16241}$, $E = \frac{25}{29} + \frac{25}{34} = \frac{1575}{986}$, $F = \frac{4}{5}$. Hisce valoribus substitutis in ultimâ expressione, & unitate pro R, invenies aream esse 785398187. Justus est hic numerus in septimâ figurâ, in octavâ verum superans Binario.

Si undecim Ordinatæ non dent aream satis exactam, erige plures; & concipe aream divisam esse in plures partes, quarum quamque seorsum quærens habebis pro lubitu justam.

Valor ipsius $\overline{1+Q}|^{n}$ exprimi potest per quamcunque trium serierum sequentium.

$$\overline{1+Q}|^{n} = 1 +$$
$$Q \times \frac{n}{1} +$$
$$Q^2 \times \frac{n}{1} \times \frac{n-1}{2} +$$
$$Q^3 \times \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} +$$
$$Q^4 \times \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} +$$
$$Q^5 \times \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} + \&c.$$

Vel $\overline{1+Q}|^{n} = 1 +$
$$R \times \frac{n}{1} +$$
$$R^2 \times \frac{n}{1} \times \frac{n+1}{2} +$$
$$R^3 \times \frac{n}{1} \times \frac{n+1}{2} \times \frac{n+2}{3} +$$

$R^4 \times$

$$R^4 \times \frac{n}{1} \times \frac{n+1}{2} \times \frac{n+2}{3} \times \frac{n+3}{4} +$$

$$R^5 \times \frac{n}{1} \times \frac{n+1}{2} \times \frac{n+2}{3} \times \frac{n+3}{4} \times \frac{n+4}{5} + \&c.$$

posito scilicet $R = \frac{1+Q}{Q}$. Vel

$$\overline{1+Q}|^n = 1 +$$

$$\frac{2+\overline{n+1}\times Q}{\overline{1+Q}|^1} \times Q \times \frac{n}{1.2} +$$

$$\frac{4+\overline{n+2}\times Q}{\overline{1+Q}|^2} \times Q^3 \times \frac{n}{1.2} \times \frac{nn-1}{3.4} +$$

$$\frac{6+\overline{n+3}\times Q}{\overline{1+Q}|^3} \times Q^5 \times \frac{n}{1.2} \times \frac{nn-1}{3.4} \times \frac{nn-4}{5.6} +$$

$$\frac{8+\overline{n+4}\times Q}{\overline{1+Q}|^4} \times Q^7 \times \frac{n}{1.2} \times \frac{nn-1}{3.4} \times \frac{nn-4}{5.6} \times \frac{nn-9}{7.8} +$$

$$\frac{10+\overline{n+5}\times Q}{\overline{1+Q}|^5} \times Q^9 \times \frac{n}{1.2} \times \frac{nn-1}{3.4} \times \frac{nn-4}{5.6} \times \frac{nn-9}{7.8} \times \frac{nn-16}{9.10}$$

$$+ \&c.$$

Primæ duæ Series demonstrantur per Casum primum Propositionis Nam si $\overline{1+Q}|^0$, $\overline{1+Q}|^1$, $\overline{1+Q}|^2$, $\overline{1+Q}|^3$, $\overline{1+Q}|^4$, &c. designent Ordinatas totidem æquidistantes in Parabolicâ figurâ. erit $\overline{1+Q}|^n$ ejusdem Ordinata, cujus distantia à $\overline{1+Q}|^0$ est n. Et sic prodit Series prima. At si in alia Parabola $\overline{1+Q}|^0$, $\overline{1+Q}|^{-1}$, $\overline{1+Q}|^{-2}$, $\overline{1+Q}|^{-3}$, $\overline{1+Q}|^{-4}$, &c. sint æquidistantes Ordinatæ, erit $\overline{1+Q}|^n$ Ordinata in eadem, cujus distantia à $\overline{1+Q}|^0$ est $-n$; sic proveniet Series secunda Sit jam in tertia Parabola &c. $\overline{1+Q}|^{-4}$, $\overline{1+Q}|^{-3}$, $\overline{1+Q}|^{-2}$, $\overline{1+Q}|^{-1}$, $\overline{1+Q}|^0$, $\overline{1+Q}|^1$, $\overline{1+Q}|^2$, $\overline{1+Q}|^3$, $\overline{1+Q}|^4$, &c. Series Ordinatarum

 æqui-

æquidistantium hinc inde progrediens in infinitum, eritque in eadem $\overline{1+Q}|^{n}$ Ordinata, distantiâ n à termino medio $\overline{1+Q}|^{0}$ remota. Et sic provenit Series tertia per Casum Secundum Propositionis. Prima abrumpit quando est n integer & affirmativus, secunda quando est n integer & negativus, & tertia in casu utroque abrumpit. Per harum quamque radices numerales commode evolvuntur in Series. Tertia reliquis multo citius convergit: ejus terminus secundus adhiberi potest pro correctione, ubi fit extractio per repetitionem calculi.

Halleius in sua methodo construendi Logarithmos, ex prima harum serierum demonstrat Seriem *Mercatoris* pro Quadratura Hyperbolæ. Sit ejus Ordinata $\overline{1+z}|^{-1}$, vel $\overline{1+z}|^{n-1}$, existente n numero infinite parvo; unde per methodos Quadrandi, area quæ jacet supra Abscissam z, id est, Logarithmus numeri $1+z$, erit $\frac{\overline{1-z}|^{n}-1}{n}$. Est vero per primam Seriem $\overline{1+z}|^{n} = 1 + \frac{n}{1}z + \frac{n}{1}\times\frac{n-1}{2}z^{2} + \frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}z^{3} +$ &c. adeoque in casu præsente, ubi est n infinite parvus, est $\overline{1+z}|^{n} = 1 + \frac{n}{1}z - \frac{n}{2}z^{2} + \frac{n}{3}z^{3} - \frac{n}{4}z^{4}$ &c. quo substituto in valore areæ, ea prodit $z - \frac{1}{2}z^{2} + \frac{1}{3}z^{3} - \frac{1}{4}z^{4} + \frac{1}{5}z^{5} -$ &c. quæ est Series *Mercatoris*.

Similiter per Seriem secundam prodit hæc regula; Sit datus numerus $1+z$, pone $R = \frac{z}{1+z}$, eritque ejus Logarithmus $R + \frac{1}{2}R^{2} + \frac{1}{3}R^{3} + \frac{1}{4}R^{4} + \frac{1}{5}R^{5} +$ &c.

Per Seriem tertiam provenit sequens regula. Sit quilibet numerus R, pone $z = \frac{\overline{R-1}|^{2}}{2R}$, eritque ejus Logarithmus

garithmus $\frac{RR-1}{2R} - \frac{1}{3}Az - \frac{2}{5}Bz - \frac{3}{7}Cz - \frac{4}{9}Dz - \frac{5}{11}Ez$ — &c. Ubi $A, B, C, D, E,$ &c. more *Newtoniano* designant terminos Seriei sicut ab initio. Hæc Series, ut ea ex qua deducitur, reliquis duabus multis vicibus celerius approximat: estque eadem generalius expressa quam, ex fundamento haud absimili, pro inventione Logarithmi Binarii prius dedimus.

Methodus inveniendi valores Serierum Arithmeticarum utcunque tarde convergentium.

In aliquibus Seriebus summa terminorum haberi nequit nisi ad paucissima figurarum loca, dummodo præter simplicem eorum additionem aliæ artes non adhibeantur. Proponatur jam Series quælibet cujus termini omnes iisdem signis afficiuntur, & quorum proximi continue tendunt esse inter se æquales; quales sunt sequentes $\frac{1}{1 \cdot 2} + \frac{1}{3 \cdot 4} + \frac{1}{5 \cdot 6} + \frac{1}{7 \cdot 8} +$ &c. $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} +$ &c. Collige summam aliquot terminorum sub initio, ii proxime addendi sint $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta,$ &c. In numeris proximis sit $r = \frac{\alpha\gamma - \beta\beta}{\alpha\beta - 2\alpha\gamma + \beta\gamma}$, & quantitatum $\alpha \times \frac{\alpha + r\beta}{\alpha - \beta}$, $\alpha + \beta \times \frac{\beta + r\gamma}{\beta - \gamma}$, $\alpha + \beta + \gamma \times \frac{\gamma + r\delta}{\gamma - \delta}$, $\alpha + \beta + \gamma + \delta \times \frac{\delta + r\varepsilon}{\delta - \varepsilon}$, $\alpha + \beta + \gamma + \delta + \varepsilon \times \frac{\varepsilon + r\zeta}{\varepsilon - \zeta}$, &c. differentiæ sint $a, b, c, d, e,$ &c. Deinde in numeris proximis sit $s = \frac{ac - bb}{ab - 2ac + bc}$, & ipsorum $a \times \frac{a + sb}{a - b}$, $a + b \times \frac{b + sc}{b - c}$, $a + b + c \times \frac{c + sd}{c - d}$, $a + b + c + d \times \frac{d + se}{d - e}$, &c. differentiæ sint $A, B, C, D,$ &c. & sit $t = \frac{AC - BB}{AB - 2AC + BC}$: atque sic procede

cede quoad libuerit. Tum erit $\alpha + \beta + \gamma + \delta + \varepsilon$ $+$ &c. $= \alpha \times \frac{\alpha + r\beta}{\alpha - \beta} + a \times \frac{a + sb}{a - b} + A \times \frac{A + tB}{A - B} +$ &c. atque ultra duos primos terminos hujus novæ Seriei raro opus erit progredi.

Ut si desideretur valor Seriei $\frac{1}{1.2} + \frac{1}{3.4} + \frac{1}{5.6} + \frac{1}{7.8}$ $+$ &c. collige primos 21 terminos, quorum summam reperio fore 6813,8410,1885. Termini proxime addendi sunt $\alpha = ,0005,2854,1226$, $\beta = ,0004,8309,1787$. $\gamma = ,0004,4326,2411$, $\delta = ,0004,0816,3265$, &c. Hinc fit $r = 1$ proxime, & $\alpha \times \frac{\alpha + r\beta}{\alpha - \beta} = ,0117,6449,6282$, $a = -,0000,0017,5096$, $b = -,0000,0014,7410$, $c = -,0000,0012,4986$, &c. Unde $s = \frac{1}{3}$ prope, & $a \times \frac{a + sb}{a - b} = -,0000,0141,8111$, quem propter signum negativum subduco ab $\alpha \times \frac{\alpha + r\beta}{\alpha - \beta}$, & remanet, 0117,6307, 8171: hic additus summæ primo inventæ 6813,8410, 1885, dat pro summa totius Seriei numerum 6931, 4718,0056, qui justus est in nonâ decimali; at ante duas hasce correctiones summa erat justa in primâ figura solâ. Si animus sit propius scopum attingere, pergendum erit ad approximationes sequentes. Si termini Seriei diversa habeant signa, conjungendi sunt, ut omnes eadem tandem habeant, ut in Serie $1 - \frac{1}{3} + \frac{1}{5}$ $- \frac{1}{7} + \frac{1}{9} -$ &c. conjunctis terminis ea evadit $\frac{2}{1.3} +$ $\frac{2}{5.7} + \frac{2}{9.11} + \frac{2}{13.15} +$ &c. Sed hic notandum est quod differentiæ a, b, c, d, e, &c. ut & A, B, C, D, &c. colligi debent subducendo quantitates antecedentes de subsequentibus. Et in omnibus hujusmodi Seriebus si p, q, r, repræsentent tres terminos ordine sequentes, p primum,

mum, q ſecundum, r tertium, & rectangulum $\frac{p+r}{2} \times q$ non ſit majus pr, valor Seriei erit infinite magnus: at magnitudinis ſemper finitæ ubi accidit contrarium. Poteſt hæc regula nonunquam fallere, ubi termini p, q, r parum diſtant ab initio Seriei, at ſi conſiſtant inter eos ab initio aliquantum remotos, evadet regula certiſſima.

Ad alia Serierum genera debent aliæ regulæ adhiberi. Sit Series regularium Polygonorum Circulo Incriptorum, exiſtente Radio unitate.

$H =$	2,0000,0000,0000,000	4
$G =$	2,8284,2712,4746,190	8
$F =$	3,0614,6745,8920,718	16
$E =$	3,1214,4515,2258,051	32
$D =$	3,1365,4849,0545,938	64
$C =$	3,1403,3115,6954,752	128
$B =$	3,1412,7725,0932,772	256
$A =$	3,1415,1380,1144,299	512

Dicatur jam ultimum Polygonum A, penultimum B, antepenultimum C, & reliqua in ſuo ordine retrorſum D, E, F, &c. atque area Circuli quæſita erit $A + \frac{A-B}{3} + \frac{4A-5B+C}{3 \cdot 15} + \frac{64A-84B+21C-D}{3 \cdot 15 \cdot 63} + \frac{4096A-5440B+1428C-85D+E}{3 \cdot 15 \cdot 63 \cdot 255} +$ &c. Ubi ſi pro A, B, C, D, E, &c. ſcribantur proprii valores, primi quatuor termini dabunt 3,1415,9265,3589,790 pro area Circuli. Hæc autem Series eſt generalis, ex natura Circuli neutiquam dependens: applicabilis eſt quotieſcunque numerorum approximantium differentiæ priores ſunt poſteriorum quaſi quadruplæ. Factores in Denominatoribus ſunt dignitates integræ numeri 4 unitatibus mi-

nutæ:

nuræ: quibus datis, coefficientes literarum in diversis terminis formantur ex multiplicatione continua numerorum 1, $\frac{n}{3}$, $\frac{n-3}{15}$, $\frac{n-15}{63}$, $\frac{n-63}{255}$, &c. Ubi pro n substituendus est ultimus Factorum in Denominatore.

Ultima quantitatum $x-1$, $2\sqrt[2]{x}-2$, $4\sqrt[4]{x}-4$, $8\sqrt[8]{x}-8$, $16\sqrt[16]{x}-16$, &c. æqualis est Logarithmo numeri x. Pro x scribe 2, & per repetitam extractionem radicis quadratæ exibunt numeri

$$
\begin{aligned}
M &= 1{,}0000{,}0000{,}0000{,}0000.\\
L &= 8284{,}2712{,}4746{,}1901.\\
I &= 7568{,}2864{,}0010{,}8843.\\
H &= 7240{,}6186{,}1322{,}0613.\\
G &= 7083{,}8051{,}8838{,}6214.\\
F &= 7007{,}0875{,}6931{,}7337.\\
E &= 6969{,}1430{,}7308{,}8294.\\
D &= 6950{,}2734{,}2438{,}7611.\\
C &= 6940{,}8641{,}2851{,}8363.\\
B &= 6936{,}1658{,}4759{,}4014.\\
A &= 6933{,}8182{,}9699{,}9493.
\end{aligned}
$$

Dicatur ultimus numerorum A, penultimus B, & sic retro, atque Logarithmus quæsitus erit $A+\frac{A-B}{1}+\frac{2A-3B+C}{1\cdot 3}+\frac{8A-14B+7C-D}{1\cdot 3\cdot 7}+\frac{64A-120B+70C-15D+E}{1\cdot 3\cdot 7\cdot 15}$ + &c. Primi quinque termini dant 6931,4718,0559, 9457 pro Logarithmo Hyperbolico Binarii. Et quomodo hæc Series procedit in infinitum facile colligitur ex eo quod de priore diximus: estque etiam universalis, proprietates Hyperbolæ minime respiciens.

Extenditur quoque Methodus hæcce Differentialis ad Resolutionem Æquationum & alia quamplurima quorum hic non fit mentio. Continetque fundamenta Serierum generalissima; ut in Reductione Æquationum Irrationalium & Fluxionalium brevi forsan monstrabo.

IV. *An*